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Product Of Weighted Composition And Differentiation Operators Between Weighted Bergman-Nevanlinna And Zygmund Type Spaces

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Abstarct: Let φ be a holomorphic self map of $\mathbb{D}$ and let ψ be an analytic function on the open unit disc $\mathbb{D}$. Let $W_{\psi,\varphi}, D$ be the weighted composition operator and differentiation operators respectively. In this paper we consider the product operator $W_{\psi,\varphi}D$ from weighted Bergman Nevanlinna spaces to Zygumund spaces and we have discussed the boundedness and compactness of this operator from weighted Bergman Nevanlinna spaces to Zygmund spaces.
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I. Introduction

Throughout the paper, ϕ denotes the non constant analytic self map of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}, \partial\mathbb{D}$ the boundary of $\mathbb{D}$. Associated with ϕ is the composition operator C_ϕ defined by $C_\phi f = f o\phi$ for $f \in H(\mathbb{D})$, the class of all analytic functions on $\mathbb{D}$. It is interesting to provide the function theoretic characterization of ϕ when ϕ induce a bounded and compact composition operators on various spaces see [3],[7],[10].

By Z, we denote the class of all $f \in H(\mathbb{D}) \cap \mathbb{C}(\overline{\mathbb{D}})$ such that

$$||f||_Z = sup|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|h < \infty(1)$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$.

From a Theorem of Zygmund see [2, Theorem 5.3] and the closed Graph theorem we see that $f \in Z$ if and only if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty.$$

It is easy to see that Z is a Banach space under the norm $||f||_z$ where

$$||f||_z = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|(2)$$

We call Z the Zygmund space.

By Z_0. we denote the little Zygmund space of $\mathbb{D}$ and is the closed subspace of Z consisting of functions f with

$$\lim_{|z|\to 1}(1-|z|^2)|f''(z)| = 0(3)$$

Let $-1 < \alpha < \infty$, the weighted Bergman Nevanlinna-space A_N^α consists of all complex valued holomorphic functions on $\mathbb{D}$ such that

$$||f||_{A_N^\alpha} = \int_{\mathbb{D}} log(1 + |f(z)|)dv_\alpha(z) < \infty$$

where

$$dv_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z), dA(z) = dxdy\pi, z = x + iy$$

In fact, $||f||_{A_N^\alpha}$ fails to be a norm on A_N^α, but $\langle f, g\rangle \to ||f-g||_{A_N^\alpha}$ defines a translation invariant metric on A_N^α and this turns A_N^α into a complete metric space. Also by the subhormonicity of f

$$log(1 + |f(z)|) \le c_\alpha ||f||_{A_N^\alpha}(1-|z|^2)^{\alpha+2} \forall f \in A_N^\alpha(4)$$

For more detail, about weighted Bergman Nevanlinna space, we refer [3] and references therein. Recently there are some papers that reveals about the boundedness and compactness of composition operators from a particulars domain space of holomorphic functions to Zygmund and Bloch spaces. See, for example, [4], [5], [6], [7], [8], [9], [10] and reference therein. In this paper, we made an attempt to discuss the boundedness and compactness of the operator $W_{\psi,\varphi}D$ from weighted Bergman Nevanlinna spaces to Zygmund type spaces by using test function technique.

II. Boundedness and Compactness

In this section, we characterize the boundedness and compactness of operators $W_{\psi,\varphi}D$ from weighted Bergman Nevanlinna spaces to Zygmund type spaces.

Recall that a linear map $T: A_N^\alpha \to X$ (Banach space) is bounded if $T(E) \subset X$ is bounded for every bounded subset E of A_N^α. We say that T is compact if $T(E) \subset X$ is relatively compact for every bounded set $E \subset A_N^\alpha$.

The following criterion for compactness is a useful tool to us and it follows from a stanard arguments, for example to those outlined in Proposition 3.11 of [1].

Lemma 1 *Let $\alpha \in (-1,\infty)$. Then a linear operator $T: A_N^\alpha \to Z(Z_0)$ is compact if and only if for any sequence $\{f_n\}$ in A_N^α with $sup_n ||f_n||_{A_N^\alpha} = M < \infty$ which converges to zero locally uniformaly on $\mathbb{D}$ we have*

$$\lim_{n\to\infty} ||Tf_n||_X \to 0 \quad in \quad X.$$

Theorem 1 *Let $\alpha > -1, \psi \in H(\mathbb{D})$ and ϕ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) $W_{\psi,\varphi}D$ maps A_N^α boundedly into Z

(ii) $W_{\psi,\varphi}D$ maps A_N^α compactly into Z.

(iii) For all $c > 0$,

$$\lim_{|\phi(z)|\to 1} (1-|z|^2).|\psi''(z)|(1-|\phi(z)|^2)exp[c(1-|\phi(z)|^2)^{\alpha+2}] = 0$$

$$\lim_{|\phi(z)|\to 1} (1-|z|^2)(\psi(z).(\phi'(z))^2|(1-|\phi(z)|^2)^3 exp[c(1-|\phi(z)|^2)^{\alpha+2}] = 0$$

$$\lim_{|\phi(z)|\to 1} (1-|z|^2).|\psi'(z).\phi''(z)+2\phi'(z).\psi'(z)|(1-|z|^2)^2 exp[c(1-|\phi(z)|^2)^{\alpha+2}] = 0$$

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi''(z)| < \infty, \sup_{z\in\mathbb{D}}(1-|z|^2)|\psi(z).(\phi'(z))^2| < \infty$$

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|2\phi'(z).\psi'(z)+\psi'(z).\phi''(z)| < \infty$$

Proof. It suffices to check two implications

$(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$,

$(i) \Rightarrow (iii)$: Suppose that (i) holds.

By taking $f(z) = z$ in A_N^α we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|(W_{\psi,\varphi}Df)''(z)| < \infty$$

But

$$|(W_{\psi,\varphi}Df)''(z)| = |\{2\psi'(z).\phi'(z)+\psi(z).\phi''(z)\}.f''(\phi(z))$$

$$+\psi(z).(\phi'(z))^2.f'''(\phi(z)).+\psi''(z).f'(\phi(z))|$$

and $f'(z) = 1, f''(z) = 0, f'''(z) = 0$.

Therefore we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi''(z) < \infty$$

Again, by taking $f(z) = z^2 2$ in A_N^α we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|2\psi'(z).\phi'(z)+\psi(z).\phi''(z)\}f''(\phi(z))+\psi(z).(\phi'(z))^2.f'''(\phi(z))|+$$

$\psi''(z).f'(\phi(z))| < \infty$

Since $|\phi(z) < 1$ and $\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi''(z)| < \infty$ and $f''(z) = 1. f'''(z) = 0.$

Therefore we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|2\psi'(z).\phi'(z)+\psi(z).\phi'(z)| < \infty.$$

Finally, taking $f(z) = z^3 6$ in A_N^α we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\{2\psi'(z).\phi'(z)+\psi(z).\phi'(z)\}f''(\phi(z))+\psi(z).(\phi'(z))^2.f'''(\phi(z))|+$$

$\psi''(z).f'(\phi(z))| < \infty$

Since $f'(z) = z^2 2$ and $f''(z) = z, f'''(z) = 1$ and $|\phi(z)| < 1$.

Therefore we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|2\psi'(z).\phi'(z)+\psi(z).\phi'(z)+\psi''(z).(\phi(z))^2 2|<\infty$$

But

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|2\psi'(z).\phi'(z)+\psi(z).\phi'(z)|<\infty$$

and $|(\phi(z))^2 2|<1$.

Therefore we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi''(z)|<\infty$$

Consider the function

$$f_\lambda(z)=\{(7(\alpha+2)+1).(1-|\phi(\lambda)|^2)^{2(\alpha+2)}(1-\overline{\phi(\lambda)}z)^{2(\alpha+2)}-[12(\alpha+2)+2].(1-|\phi(\lambda)|^2)^{2(\alpha+2)}(1-\overline{\phi(\lambda)}z)^{3(\alpha+2)}$$

$$+5((\alpha+2)+1).(1-|\phi(\lambda)|^2)^{3(\alpha+2)}(1-\overline{\phi(\lambda)}z)^{4(\alpha+2)}\}$$

$$\times\, exp[6c\{12(1-|\phi(\lambda)|^2)^{\alpha+2}(1-\overline{\phi(\lambda)}z)^{2(\alpha+2}-13(1-|\phi(\lambda)|^2)^{2(\alpha+2)}(1-\overline{\phi(\lambda)}z)^{3(\alpha+2)}\}]$$

Also one can easily check that

$$f_\lambda'(\phi(z))=2(\alpha+2)\overline{(\phi(\lambda)}(1-|\phi(\lambda)|^2)^{\alpha+3}exp[c(1-|\phi(\lambda)|^2)^{\alpha+2}]$$
$$f_\lambda''(\phi(z))=0$$
$$f_\lambda'''(\phi(z))=0$$

Since $W_{\psi,\varphi}D: A_N^\alpha\to Z$ is bounded.

Therefore there exists a constant $M_3>0$ such that

$$M_1\geq(1-|\lambda|^2)|\{2\psi'(z).\phi'(z)+\psi(z).\phi''(z)\}f''(\phi(z))+\psi(z).(\phi'(z))^2.f'''(\phi(z))+\psi''(z).f'(\phi(z))|$$

Hence

$$(1-|\lambda|^2).2(\alpha+2)|\overline{(\phi(\lambda)}|.(\psi''(z)|(1-|\phi(\lambda)|^2)^{\alpha+3}exp[c(1-|\phi(\lambda|^2)^{\alpha+2}]\leq M_1$$

This implies that

$$(1-|\lambda|^2)|\psi''(z)|(1-|\phi(\lambda)|^2)exp[c(1-|\phi(\lambda|^2)^{\alpha+2}]\leq M_1(\alpha+2)|\overline{\phi(\lambda}|.(1-|\phi(\lambda)|^2)^{\alpha+2}$$

Taking the limit $|\phi(\lambda)|\to 1$ on both sides in the above inequality we get

$$\lim_{|\phi(\lambda)|\to 1}(1-|\lambda|^2)|\psi''(z)|(1-|\phi(\lambda)|^2)^2exp[c(1-|\phi(\lambda)|^2)^{\alpha+2}]=0$$

Again, consider the function

$$g_\lambda(z)=\{(4(\alpha+2)+4).(1-|\phi(\lambda)|^2)^{\alpha+2}(1-\overline{\phi(\lambda)}z)^{2(\alpha+2)}$$

Also, one can easily check that

$$g_\lambda'(\phi(z))=0$$
$$g_\lambda''(\phi(z))=0$$
$$g_\lambda'''(\phi(z))=c(1-|\phi(\lambda)|^2)^{\alpha+5}exp[c(1-|\phi(\lambda)|^2)^{\alpha+2}]$$

Since $W_{\psi,\varphi}D: A_N^\alpha\to Z$ is bounded.

Therefore there exists a constant $M_2>0$ such that

$$M_2\geq(1-|\lambda|^2)|\{2\psi'(z).\phi'(z)+\psi(z).\phi''(z)\}g''(\phi(z))+\psi(z).(\phi'(z))^2.g'''(\phi(z))+\psi''(z).g'(\phi(z))|$$

Hence

$$(1-|\lambda|^2).|\psi(z).(\phi'(z))^2|.C(1-|\phi(\lambda)|^2)^{\alpha+5}exp[c(1-|\phi(\lambda|^2)^{\alpha+2}]\leq M_2$$

This implies that

$$(1-|\lambda|^2)|\psi(z).\phi'(z))^2|(1-|\phi(\lambda)|^2)^3exp[c(1-|\phi(\lambda|^2)^{\alpha+2}]\leq M_2.(1-|\phi(\lambda)|^2)^{\alpha+2}$$

Taking the limit $|\phi(\lambda)|\to 1$ on both sides in the above inequality we get

$$\lim_{|\phi(\lambda)|\to 1}(1-|\lambda|^2)|\psi(z).(\phi'(z))^2|(1-|\phi(\lambda)|^2)^3exp[c(1-|\phi(\lambda)|^2)^{\alpha+2}]=0$$

Finally, let $\lambda\in\mathbb{D}$ and $c>0$, consider the function

$$h_\lambda(z)=\{(3+2(\alpha+2))1-|\phi(\lambda)|^2)^{\alpha+2}(1-\overline{\phi(\lambda)}z)^{2(\alpha+2)}-(4(\alpha+2)+4).1-|\phi(\lambda)|^2)^{\alpha+3}(1-\overline{\phi(\lambda)}z)^{2(\alpha+2)+1}$$

$$+8(2(\alpha+2)+1).(1-|\phi(\lambda)|^2)^{\alpha+4}(1-\overline{\phi(\lambda)}z)^{2(\alpha+2)}$$

$$\times\{c2(4(\alpha+2)^2+6(\alpha+2)+2).(1-|\phi(\lambda)|^2)^{(\alpha+2)}(1-\overline{\phi(\lambda)}z)^{2(\alpha+2)}-(8(\alpha+2)^2+8(\alpha+2)).(1-|\phi(\lambda)|^2)^{2(\alpha+3)}(1-\overline{\phi(\lambda)}z|^{2(\alpha+2)+1)}$$

$$+(4(\alpha+2)^2+2(\alpha+2))1-|\phi(\lambda)|^2)^{\alpha+4}(1-\overline{\phi(\lambda)}z)^{2(\alpha+2)+2}\}$$

Also, one can easily check that

$$h_\lambda'(\phi(z))=0$$

$$h_\lambda''(\phi(z)) = c(1-|\phi(\lambda)|^2)^{\alpha+2} exp[c(1-|\phi(\lambda)|^2)^{\alpha+2}]$$
$$h_\lambda'''(\phi(z)) = 0$$

Since $W_{\psi,\varphi}D: A_N^\alpha \to Z$ is bounded.

Therefore there exists a constant $M_3 > 0$ such that

$$M_3 \geq (1-|\lambda|^2)|\{2\psi'(z).\phi'(z) + \psi(z).\phi''(z)\}h''(\phi(z)) + \psi(z).(\phi'(z))^2.h'''(\phi(z)) + \psi'''(z).h'(\phi(z))|$$

Hence

$$(1-|\lambda|^2).|2\psi'(z).\phi'(z) + \psi(z).\phi''(z)|.C(1-|\phi(\lambda)|^2)^{\alpha+2} exp[c(1-|\phi(\lambda|^2)^{\alpha+2}] \leq M_3$$

$$\Rightarrow \quad (1-|\lambda|^2)|2\phi'(z).\psi'(z) + \psi(z).\phi''(z)|(1-|\phi(\lambda)|^2)^2 exp[c(1-|\phi(\lambda|^2)^{\alpha+2}] \leq M_3 C.(1-|\phi(\lambda)|^2)^\alpha$$

Taking the limit $|\phi(\lambda)| \to 1$ on both sides in the above inequality we get

$$\lim_{|\phi(\lambda)|\to 1} (1-|\lambda|^2)|2\phi'(z)|.\psi'(z) + \psi(z).\phi''(z)|(1-|\phi(\lambda)|^2)^2 = 0$$

$(iii) \Rightarrow (ii)$ Assume that the conditions in (iii) is valid for all $c > 0$

Note that if $f \in A_N^\alpha$, then by (1.1) and Cauchy integral formula for derivatives, we have

$$(1-|z|^2)|f'(z)| \leq 2\pi \int_{\delta D} |f(z + 12(1-|z|)\xi)|.|d\xi|$$
$$\leq exp[c_\alpha ||f||_{A_N^\alpha}(1-|z|^2)^{\alpha+2}]$$

Choose for $M > 0$ any sequence $\{f_n\}$ in A_N^α such that

$$||f_n||_{A_N^\alpha} \leq M \quad and \quad f_n \to 0$$

locally uniformly on $\mathbb{D}$.

Then for each $r \in (0,1)$ we have $\sup_{|z|\leq r}(1-|z|^2)|W_{\psi,\varphi}Df_n)''(z)|$

$$= \sup_{|z|\leq r}(1-|z|^2)|\{2\psi'(z).\phi'(z) + \psi(z).(\phi''(z))\}$$
$$f_n''(\phi(z)) + \psi(z)).(\phi'(z))^2.f_n'''(\phi(z)) + \psi'''(z)).f_n'(\phi(z))|$$
$$\leq A.\sup_{|z|\leq r}|f_n''(\phi(z))| + B.\sup_{|z|\leq r}|f'''(\phi(z))| + C.\sup_{|z|\leq r}|f_n'(\phi(z))|$$

where

$$A = 2(\psi'(z)).(\phi'(z)) + \psi(z)\phi''(z)$$

$$B = \psi(z)\phi'(z))^2$$

$$C = |\psi'''(z)|$$

On the other hand, whenever $r \to 1$ we have $\sup_{|z|>r}|W_{\psi,\varphi}Df_n)''(z)|$

$$= \sup_{|z|>r}(1-|z|^2)|\{2\psi'(z).\phi'(z) + \psi(z).(\phi''(z))\}$$
$$+\psi(z).(\phi'(z))^2.f_n'''(\phi(z)) + \psi'''(z)).f_n'(\phi(z))|$$
$$\leq \sup_{|z|>r}(1-|z|^2|2\psi'(z).\phi'(z) + \psi(z).(\phi''(z)|.|f_n''(\phi(z))|$$
$$+.\sup_{|z|>r}|\psi(z).(\phi'(z))^2|.|f_n'''(\phi(z))|$$
$$+.\sup_{|z|>r}|\psi'''(z)|.|f_n'(\phi(z))|$$
$$= \sup_{|\phi(z)|\to r}(1-|z|^2).|2\psi'(z).\phi'(z) + \psi(z)\phi''(z)|(1-|\phi(z)|^2)^2 exp[c_\alpha(1-|\phi(z)|^2)^{\alpha+2}]$$
$$+ \sup_{|\phi(z)|\to r}(1-|z|^2).|(\psi(z).(\phi'(z))^2|.48(1-|\phi(z)|^2)^3 exp[c_\alpha(1-|\phi(z)|^2)^{\alpha+2}]$$
$$+ \sup_{|\phi(z)|>r}(1-|z|^2).|\psi'''(z)|(1-|z|^2)^2 exp[c_\alpha(1-|\phi(z)|^2)^{\alpha+2}]$$
$$\to 0 \quad as \quad r \to 1$$

Combining the above estimates we see that $||W_{\psi,\varphi}D||_Z \to 0$ as $n \to \infty$. This complete the proof.

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